Lift on an Oscillating Body of Revolution

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The circulatory lift on an oscillating body of revolution may be predicted in terms of a vector circulation on a control surface surrounding the body and its wake if the net vorticity discharged into the main flow from the boundary layer is known The total lift is obtained by adding to the circulatory lift two additional components which depend on acceleration and are associated with the "apparent mass" of the body In addition, this paper reviews some kinematical relationships between the vorticity in a boundary layer and that in the wake of a body and discusses a criterion for calculating the transport of vorticity into the wake

Nomenclature

i,j,k = unit vectors
n = unit normal
p = pressure
q = velocity vector
r = radius vector
S = surface
t = time
T = transport of vorticity
$\mathbf{V}_\infty$ = freestream velocity
α,β = circulation in circuits produced by the intersection of the "control surface" with the diametral planes $\Omega = 0$ and $\Omega = \pi/2$
Γ = vector circulation $= \int_s (\mathbf{q} \times \mathbf{n})dS$
δ = boundary layer thickness
ζ = vorticity vector
μ = viscosity of fluid
ν = kinematic viscosity of fluid
ρ = density of fluid
Σ = surface
τ = period
ω = frequency of oscillation
Ω = angle of meridian from vertical plane

Introduction

A THEORETICAL prediction of the circulatory lift on a streamlined body of revolution meets with two difficulties: 1) the flow is truly three-dimensional since the thickness of the body is of the same order as its "span"; and 2) there is no criterion (such as the Kutta-Joukowski condition on a wing) that determines uniquely the intensity and distribution of vorticity in an otherwise inviscid and irrotational flow

As is well known, if a body moves with a constant velocity, the potential-flow solution predicts the existence of a destabilizing hydrodynamic moment only If the body accelerates, however, the potential-flow solution also predicts lift and drag forces These forces are in phase with the acceleration and directly proportional to it; they cease to exist if the acceleration vanishes

Because of separation of the boundary layer from the upper rear portion of the body, the actual measured value of the destabilizing moment in an incompressible fluid in steady motion is somewhat less than predicted by potential flow In addition, separation generates a lift and, of course, a drag is present Furthermore, if the motion is time-dependent, a phase lag in the build-up of the forces occurs

The circulatory lift may be predicted theoretically if the net vorticity discharged into the main body of the fluid from the boundary layer at any instant of time is known By means of a generalization of the Kutta-Joukowski theorem, the component of lift caused by a vector circulation on a control surface surrounding the body and its wake may then be found The total lift is obtained by adding to the circulatory lift two additional components which depend on acceleration and are associated with the "apparent mass" of the body

The essential difficulty, therefore, is due to the fact that viscosity must be included in the formulation of the problem, since the complicated viscous processes which take place on the surface of the body cannot be accounted for by a convenient criterion such as the Kutta-Joukowski trailing edge condition on wings

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Previous Investigations

Early attempts at predicting the circulation from detailed boundary-layer calculations were due to von Kármán and Millikan[1] and Howarth[2] Howarth's calculation of the lift on elliptic cylinders was based on the condition that, in steady flow, the total flux of vorticity into the wake must be zero, a criterion due to G I Taylor Later, Preston[3] and Spence[4] successfully applied detailed boundary layer and wake calculations to predict the circulation of airfoils; they specified that the pressures at the trailing edge shall have the same value when determined from potential flow values outside the boundary layer above and below the airfoil Sears,[5] in an illuminating discussion of the subject, showed that the Preston-Spence criterion is entirely compatible with that of Howarth He also remarked that in unsteady flow, the Preston-Spence criterion still applies if separation does not occur forward of the trailing edge Now, however, the net flux of vorticity at any given instant of time is not zero, although it vanishes in the course of a cycle Sears showed that the circulation at any time could be expressed in terms of the velocity potential of the exterior flow above and below the trailing edge of the airfoil

More recently, Nonweiler[6] and Squire[7] extended the boundary-layer approach to bodies in three-dimensional flow and obtained good agreement with experimental results Last but not least, Milne-Thomson[8] discussed with great clarity the fundamental physical and mathematical relationships which form the basis of the theories mentioned above

Force on a Body in Incompressible, Unsteady, Viscous Flow

Milne-Thomson[8] has shown that the force on a body at rest in a steady flow of a homogeneous, viscous liquid can be found in terms of velocities on a large control surface which completely surrounds the body In this paper an attempt is made to extend his method to a body which performs a small, periodic motion

The equations of motion of a viscous fluid in the absence of external forces is given by

$$\rho(d\mathbf{q}/dt) = \nabla\Phi \tag{1}$$

where Φ denotes the stress tensor Alternatively,

$$\rho(\partial\mathbf{q}/\partial t) = \nabla[\Phi - \rho(\mathbf{qq})] \tag{2}$$

Integrating over the volume between the control surface Σ and the body surface A (Fig 1) and using Gauss' theorem,

$$\int_V \rho \frac{\partial \mathbf{q}}{\partial t} dV = -\int_A \mathbf{n} \ [\Phi - \rho(\mathbf{qq})]dS + \int_\Sigma \mathbf{n} \ [\Phi - \rho(\mathbf{qq})]dS \tag{3}$$

Since the fluid is viscous, the velocity vector at the wall has no tangential component relative to it and its direction and magnitude are determined by the motion of the body Consequently

$$\int_A \mathbf{n} \ (\mathbf{qq})dS = \int_A (\mathbf{n} \ \mathbf{q})\mathbf{q}dS = 0 \tag{4}$$

Therefore, if the viscosity μ and density ρ are constant, the force $\mathbf{F}$ on the body may be obtained from

$$\mathbf{F} = \int_A \mathbf{n} \ \Phi dS = \int_V \rho \frac{\partial \mathbf{q}}{\partial t} dV + \int_\Sigma [-p\mathbf{n} - \mu(\mathbf{n} \times \boldsymbol{\zeta}) - \rho(\mathbf{n} \ \mathbf{q})\mathbf{q}]dS \tag{5}$$

In steady flow, the volume integral in Eq (5) vanishes, and the problem then consists in estimating velocities on the surface Σ which, since it is arbitrary, may be made as large as desired In unsteady flow we cannot take advantage of the simplifications resulting from this approach since, in order to evaluate the volume integral, we have to know the velocity everywhere We note, however, that in the case of streamlined bodies at small angles of attack the velocity field is well approximated by the potential flow field except in the boundary layer and in the wake On the other hand, $\mathbf{q}$ tends to its value $\mathbf{q}_\Sigma$ at a sufficiently large distance from the body It appears, therefore, reasonable to approximate the volume integral in the force equation by

$$\int_V \rho \frac{\partial \mathbf{q}}{\partial t} dV \cong \int_V \rho \frac{\partial \mathbf{q}_p}{\partial t} dV + \int_V \rho \frac{\partial \mathbf{q}_\Sigma}{\partial t} dV \tag{6}$$

where $\mathbf{q}_p$ denotes the velocity appropriate to potential flow To determine the value of $\mathbf{q}_\Sigma$, we proceed in much the same way as Milne Thomson We let Σ be a sphere so large that the velocity and pressure on its surface may be approximated by

$$\mathbf{q}_\Sigma = \mathbf{V}_\infty + \mathbf{U}_s + \mathbf{U} \tag{7}$$

$$p_\Sigma = p + p_s + p \tag{8}$$

In these equations $\mathbf{U}$ and p are small deviations of the first order from the uniform velocity $\mathbf{V}_\infty$ and pressure p and are independent of time, whereas $\mathbf{U}$ and p are time-dependent deviations Substituting in the equation of motion and using Oseen's approximation we obtain relationships for $\mathbf{U}_s$ and $\mathbf{U}$, namely

$$V_\infty \frac{\partial \mathbf{U}_s}{\partial x} = -\frac{1}{\rho}\nabla p_s + \nu\Delta\mathbf{U} \tag{9}$$

$$\frac{\partial \mathbf{U}}{\partial t} + V_\infty \frac{\partial \mathbf{U}}{\partial x} = -\frac{1}{\rho} \nabla p + \nu\Delta\mathbf{U} \tag{10}$$

The first equation contains only steady flow quantities and leads to the solution given in Ref 8 To deal with the second equation, we introduce the parameter k defined by

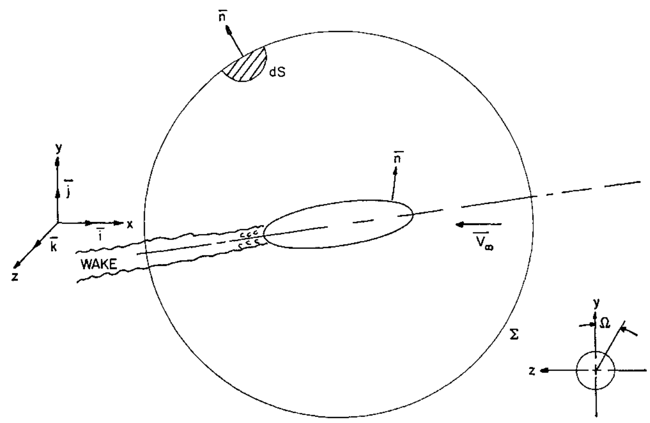

Fig 1 Coordinate system

$$k = V_\infty/2\nu$$

and obtain the equations of motion and continuity for $\mathbf{U}$

$$\frac{1}{\rho}\nabla p = \nu\left(\Delta - 2k\frac{\partial}{\partial x} - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\mathbf{U} \tag{11}$$

$$\nabla \ \mathbf{U} = 0 \tag{12}$$

It follows that the unsteady pressure p satisfies Laplace's equation In terms of a potential ϕ, which also satisfies Laplace's equation,

$$p = \rho\left(\frac{\partial\phi}{\partial t} + V_\infty \frac{\partial\phi}{\partial x}\right) \tag{13}$$

A particular solution of Eq (11) is given by

$$\mathbf{U} = \mathbf{q}_1 = -\nabla\phi \tag{14}$$

and the complete solution is of the form

$$\mathbf{U} = \mathbf{q}_1 - \mathbf{v}_2 \tag{15}$$

where $\mathbf{v}_2$ satisfies the equations

$$\left(\Delta - 2k\frac{\partial}{\partial x} - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\mathbf{v}_2 = 0 \tag{16}$$

$$\nabla \ \mathbf{v}_2 = 0 \tag{17}$$

We note that the potential ϕ may be expressed as the product of two functions, one containing space and the other time coordinates only A particular potential ϕ_{10} can, therefore, be found by the same reasoning as in steady flow

$$\phi_{10} = -(1/r) \cot\tfrac{1}{2}\theta(\alpha \cos\Omega + \beta \sin\Omega)T(t) \tag{18}$$

In this equation (r, θ, Ω) represent polar coordinates and $T(t)$ is a function of time Clearly, the velocity $\mathbf{q}_{10} = -\nabla\phi_{10}$ becomes infinite when $\theta \to 0$ and, therefore, a further component is chosen such that

$$\mathbf{v}_2 = \mathbf{q}_2 + \mathbf{v}_3 \tag{19}$$

$$\left(\Delta - 2k\frac{\partial}{\partial x} - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\mathbf{q}_2 = 0 \tag{20}$$

This velocity component can also be obtained from a potential function ψ which must satisfy Eq (20) Thus

$$\mathbf{q}_2 = -\nabla\psi \tag{21}$$

where the potential ψ again consists of the product of a space and time function Let

$$\psi(x,y,z,t) = \psi_0(x,y,z) \ T(t) \tag{22}$$

then ψ_0 must satisfy

$$[\Delta - 2k(\partial/\partial x) - \lambda]\psi_0 = 0 \quad (23)$$

or

$$(\Delta - \zeta^2)(e^{-kx}\psi_0) = 0 \quad (24)$$

where $\zeta^2 = k^2 + \lambda$ and λ is a separation constant A particular solution of Eq (24) is given by

$$e^{-kx}\psi_0 = (e^{-\zeta r}/r) \cot\tfrac{1}{2}\theta(\alpha \cos\Omega + \beta \sin\Omega) \quad (25)$$

We now make the restriction that

$$k^2 \gg |\lambda|$$

which implies

$$\zeta \cong k \quad (26)$$

This restriction does not constitute a serious limitation in practice Consider, for instance, a sinusoidal variation of $T(t)$ with time The value of $|\lambda|$ is ω/ν and if the frequency ω is not too large, our requirement is easily satisfied An alternative way of stating our restriction is that the Reynolds number

$$V_\infty\delta_0/\nu \gg 1$$

where $\delta_0 = (2\nu/\omega)^{1/2}$ denotes the "depth of penetration of shear waves" With this limitation, the sum $(\phi_{10} + \psi)$ has no infinity when $\theta = 0$ However, $\mathbf{q}_2$ does not satisfy the equation of continuity and a third velocity component $\mathbf{q}_3$ must be added such that

$$\left(\Delta - 2k\frac{\partial}{\partial x} - \frac{1}{\nu}\frac{\partial}{\partial t}\right)\mathbf{q}_3 = 0 \quad (27)$$

$$\nabla \mathbf{q}_3 = \left(2k\frac{\partial}{\partial x} + \frac{1}{\nu}\frac{\partial}{\partial t}\right)\psi \quad (28)$$

After some algebraic manipulation it can be shown that $\mathbf{q}_3$ is similar in form to its steady flow counterpart, namely,

$$\mathbf{q}_3 = -(2k/r)e^{-k(\ -x)}(\alpha\mathbf{j} + \beta\mathbf{k})T(t) \quad (29)$$

Note that $\mathbf{q}_3$ is perpendicular to the freestream velocity $\mathbf{V}_\infty$ and is negligible everywhere except in the wake The complete solution for $\mathbf{U}$ therefore consists of four components:

$$\mathbf{U} = \mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4 \quad (30)$$

where $\mathbf{q}_4$ is the complementary solution that satisfies Eq (15) and the equation of continuity.

We now return to Eq (5) for the force on the body and define the following vector quantities

$$\mathbf{F} = \mathbf{G} + \mathbf{P} + \mathbf{Q} + \mathbf{R} \quad (31)$$

where

$$\mathbf{G} = \int_V \rho\left(\frac{\partial\mathbf{q}}{\partial t}\right)dV$$

$$\mathbf{P} = -\int_\Sigma p\mathbf{n}dS$$

$$\mathbf{Q} = -\int_\Sigma \mu(\mathbf{n} \times \boldsymbol{\zeta})dS$$

$$\mathbf{R} = -\int_\Sigma \rho(\mathbf{n}\ \mathbf{q}_\Sigma)\mathbf{q}_\Sigma dS$$

The evaluation of **P**, **Q**, and **R** differs in detail, but not in principle, from the steady flow case and will not be repeated here If we introduce a vector potential **B** such that

$$\mathbf{q}_4 = \nabla \times \mathbf{B} \quad (32)$$

the unsteady force on the body finally reduces to

$$\mathbf{F} = \rho\mathbf{V}_\infty \times \mathbf{\Gamma}_3 + \mathbf{V}_\infty I + \rho\int_\Sigma\left(\mathbf{n} \times \frac{\partial\mathbf{B}}{\partial t}\right)dS + \int_V \rho\frac{\partial\mathbf{q}}{\partial t}dV - \int_\Sigma \rho\frac{\partial\phi}{\partial t}\mathbf{n}\,dS \quad (33)$$

where $\mathbf{\Gamma}_3 = \int_\Sigma (\mathbf{n} \times \mathbf{q}_3)\,dS$ denotes the unsteady vector circulation over Σ, and $I = -\rho\int_\Sigma (\mathbf{n}\ \mathbf{q}_4)dS$ represents the unsteady inflow into Σ which takes place predominantly in the wake

We now replace the volume integral in Eq (33) by its approximation given by Eq (6) and substitute for $\mathbf{q}_\Sigma$ from Eqs (7) and (30):

$$\int_V \rho\frac{\partial\mathbf{q}_\Sigma}{\partial t}dV = \int_V \rho\frac{\partial}{\partial t}(\mathbf{q}_1 + \mathbf{q}_2 + \mathbf{q}_3 + \mathbf{q}_4)dV \quad (34)$$

Applying Gauss' theorem and substituting into (33), the force on the body is given by

$$\mathbf{F} = \rho\mathbf{V}_\infty \times \mathbf{\Gamma}_3 + \mathbf{V}_\infty I + \int_V \rho\frac{\partial\mathbf{q}_3}{\partial t}dV + \int_V \rho\frac{\partial\mathbf{q}_p}{\partial t}dV \quad (35)$$

This force can be further subdivided into its unsteady lift and drag components, namely

$$\mathbf{L} = \rho\mathbf{V}_\infty \times \mathbf{\Gamma}_3 + \int_V \rho\frac{\partial\mathbf{q}_3}{\partial t}dV + \left\{\int_V \rho\frac{\partial\mathbf{q}_p}{\partial t}dV\right\} \text{ normal to } \mathbf{V}_\infty \quad (36)$$

$$\mathbf{L} = 4\pi\rho V_\infty(\alpha\mathbf{j} + \beta\mathbf{k}) + \frac{4\pi\mu}{V_\infty}\left(\frac{\partial\alpha}{\partial t}\mathbf{j} + \frac{\partial\beta}{\partial t}\mathbf{k}\right) + \left\{\int_V \rho\frac{\partial\mathbf{q}_p}{\partial t}dV\right\} \text{ normal to } \mathbf{V}_\infty \quad (37)$$

and

$$\mathbf{D} = \mathbf{V}_\infty I + \left\{\int_V \rho\frac{\partial\mathbf{q}_p}{\partial t}dV\right\} \text{ parallel to } V_\infty \quad (38)$$

The first term in Eq (36) constitutes a generalization of the Kutta-Joukowski theorem for the lift The second and third terms in Eq (36) arise as a result of the body's acceleration and are associated with the "apparent mass" of the fluid In an inviscid fluid $\mathbf{q}_3$ is, of course, zero and Eq (36) reduces to its well-known potential flow solution As in steady flow, α and β represent time-dependent circulations in distant circuits produced by the intersection of the sphere Σ with vertical and horizontal meridional planes In the case of a body of revolution, these circulations must be found from an analysis of the boundary layer

Some Kinematical Relationships

The relationship between the vorticity in a boundary layer and that in the wake of a body, in steady or unsteady motion, can be established according to the following theorem, which is quoted from Ref 8:

Theorem: Let S be a closed surface every point of which is in contact with the fluid and which cuts the wake in vortex lines Then, if the fluid velocity is finite and continuous over S, the vector circulation over S is zero

Proof: The vector circulation $\mathbf{\Gamma}$ is defined as follows:

$$\mathbf{\Gamma} = \int_s (\mathbf{n} \times \mathbf{q})dS \quad (39)$$

An alternate expressions for $\mathbf{\Gamma}$ is given by

$$\mathbf{\Gamma} = \int_s \mathbf{r}(\mathbf{n}\ \boldsymbol{\zeta})dS \quad (40)$$

Outside the wake $\boldsymbol{\zeta} = 0$ (by definition of the wake) and inside $\mathbf{n}\ \boldsymbol{\zeta} = 0$ since the vortex lines lie on S

Corollary 1: The net vorticity in any section of the wake cut off by a closed surface that intersects the wake in vortex lines is zero

Proof:

$$\mathbf{\Gamma} = \int_s (\mathbf{n} \times \mathbf{q})dS = \int_V (\nabla \times \mathbf{q})dV = \int_V \boldsymbol{\zeta}dV = 0 \quad (41)$$

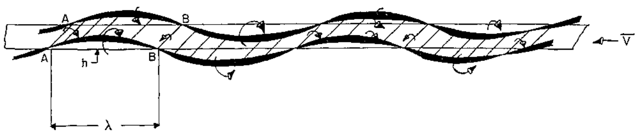

Fig 2 Vortex pattern in the wake of a body performing heaving oscillations

Corollary 2: Let S_1 be a closed surface which surrounds a body and cuts the wake in vortex lines The net vorticity in the boundary layer and that portion of the wake which lies inside S_1 is zero

Proof The vector circulation over S_1 is zero by the theorem The circulation over the body is also zero, since $\mathbf{q} = 0$ on its surface

The well-known theorem concerning the flux of vorticity in steady flow can now be easily proved:

Consequence 1 In steady flow, the net transport of vorticity from the boundary layer into the wake is zero

Proof Let S_1 be a closed surface which cuts the wake in vortex lines at time t, and let S_2 be another closed surface which cuts the wake in vortex lines at time $(t + \delta t)$ By corollary 2, the net vorticity in the wake which lies inside S_1 is equal to the net vorticity in the wake which lies inside S_2 since each is equal to the net vorticity in the boundary layer, and this does not change with time

Consequence 2 Let τ be the period of a time-varying flow Then, the net transport of vorticity from the boundary layer into the wake in the interval of time is zero

Proof At time t, let S_1 be a closed surface that surrounds the body and cuts the wake in vortex lines, and let S_2 be another such surface at time $(t + \tau)$

Since the net vorticity in the boundary layer at t and $(t + \tau)$ is the same, the net vorticity in the wake which lies inside S_1 is equal to that which lies inside S_2

It follows that the net vorticity in the wake which lies inside the closed surface $(S_2 - S_1)$ is zero

As an example, consider an imaginary, large sphere Σ such that at time t it cuts the wake along line AA in Fig 2 Then, by the theorem, the vector circulation over Σ is zero at that instant of time An interval of time $\frac{1}{2}\tau$ later, line BB will lie on Σ and the vector circulation will once again be zero

The conditions of the theorem will be violated, however, while the vortices bounded by $AABB$ move through the surface of the sphere Σ This gives rise to a vector circulation over Σ and hence to lift

Transport of Vorticity

The transport of vorticity from the boundary layer of a slender ellipsoid of revolution has been studied experimentally in great detail by Harrington[9] and, more recently by Rodgers [10] These investigations show the formation of regions of vorticity in the neighborhood of the body and the subsequent rolling up of these regions into concentrated vortices at some distance downstream A theoretical and experimental analysis of the laminar (and, to a lesser extent, also turbulent) boundary layer has been carried out by Eichelbrenner,[11] leading to the possibility of relating the vorticity in the wake to the properties of the boundary layer where this vorticity originates The measurements of Rodgers indicate that, near the separation lines predicted by Eichelbrenner, there is a sharp increase in vorticity at considerable distances from the body and that this vorticity is carried downstream by the main flow

In view of these observations in steady flow, it seems permissible to estimate the transport of vorticity as follows:

Let C be the separation line Let S_{BL} be an open surface formed by moving a normal to the body along the closed curve C and let the length of the normal be locally equal to the boundary-layer thickness δ Let $\mathbf{n}$ be a unit vector normal to S_{BL}

Near separation the transport of vorticity due to diffusion is small compared with that due to convection so that the flux of vorticity through S_{BL} is given by

$$\mathbf{T} = \int_{S_{BL}} (\mathbf{n}\ \mathbf{q})\boldsymbol{\zeta} dS \tag{42}$$

In unsteady flow, as under steady conditions, regions of vorticity form around the body in the course of a cycle The vorticity in the wake is now a function of time and, as before, depends on the net transport of vorticity from the boundary layer

Sears[5] remarked that in unsteady flow the flux of vorticity across a separation line should account for the rate of motion of this line and proposed the expression

$$\mathbf{T} = \int_{S_{BL}} (\mathbf{n}\ \mathbf{q}_{el})\boldsymbol{\zeta} dS \tag{43}$$

where $\mathbf{q}_{el}$ denotes the relative boundary-layer velocity component as seen by an observer moving with the separation line Needless to say, the problem of predicting the transport of vorticity is a difficult one An attempt to apply this theory to a slender ellipsoid of revolution and a description of certain experiments on this ellipsoid performing small heaving oscillations will be given in a subsequent paper

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